2050A Revision Exercise: 2017 1st term

1. Use the ε-$\mathbb{N}$ definition to show that $\lim \frac{n+(-1)^n}{n^2-1} = 0$.
2. Use the ε-$\mathbb{N}$ definition to show that $\lim_n \left(\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{(n-1)\cdot n}\right) = 1$.
3. Using the definition show that the sequence $(\frac{n^2+1}{2n+1})$ diverges to ∞.
4. Show that if $x_n > 0$ and $\lim x_n = a$, then $\sqrt{x_n} \to \sqrt{a}$.
5. Suppose that $x_1 > y_1 > 0$ and $x_{n+1} x_n y_n$ and $y_{n+1} = \frac{x_n + y_n}{2}$. Show that $\lim x_n$ and $\lim y_n$ exist, moreover, $\lim x_n = \lim y_n$.
6. Show that if $\lim x_n = a$ exists, then $\lim \frac{x_1 + \cdots + x_n}{n} = a$.
7. Show that if (x_n) is an unbounded sequence, then there is a subsequence (x_{n_k}) diverges to ∞.
8. Suppose that (x_n) is an unbounded sequence and does not diverges to ∞. Show that there are two subsequences (x_{n_k}) and (x_{m_k}) of (x_n) such that (x_{n_k}) diverges to ∞ and $\lim_k x_{m_k}$ exists.
9. Suppose that $|r| < 1$ and (a_n) is bounded. Let $x_n := \sum_{k=0}^n a_k r^k$. Show that the sequence (x_n) is convergent.
10. Using the definition, show that $\lim_{x\to -1} \frac{x-3}{x^2-9} = \frac{1}{2}$; $\lim_{x\to\infty} \frac{x-1}{x+2} = 1$ and $\lim_{x\to\infty} \frac{x^2+x}{x+1} = \infty$.
11. Let $x \in [0,1]$ and $f(x) = 0$ if $x \in \mathbb{Q}$; otherwise, $f(x) = 0$. Find the right and left limits of f at $x = 1/2$.
12. Show that $\lim_{x\to\infty} f(x) = L$ exists if and only if for any sequence (x_n) with $x_n \to \infty$, we have $f(x_n) \to L$, where $L \in \mathbb{R}$ or $L = \infty$.
13. Let f be a function defined on $[a,b]$. Suppose that $\lim_{x\to c\pm} f(x)$ both exist for all $c \in [a,b]$. Show that f is bounded.
14. If f and g are continuous functions on $\mathbb{R}$, show that the function $h(x) := \max(f(x), g(x))$ for $x \in [a,b]$ is also continuous.
15. Let f be a continuous function defined on $[a,b]$. Let $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ be any partition on $[a,b]$. Show that there is $\xi \in [a,b]$ such that $f(\xi) = \sum_{k=0}^n \frac{f(x_0)+\cdots+f(x_n)}{n+1}$.
16. Show that if f is a continuous strictly positive function on $[a,b]$, then $\frac{1}{f(x)}$ is also continuous on $[a,b]$.
17. Prove by the definition that the functions $f(x) = x^{1/3}$ is uniformly continuous on $[0,1]$ and $g(x) = \sin x^2$ is not uniformly continuous on $\mathbb{R}$.
18. Is the function $f(x) = x^2$ uniformly continuous on $\mathbb{R}$?
19. Is the function $f(x) = \frac{\sin x}{x}$ uniformly continuous on $(0,\pi)$?
20. Let f be a continuous function defined on $[a,\infty)$. Show that if $\lim_{x\to\infty} f(x)$ exists, then f is uniformly continuous on $[a,\infty)$. Is the converse true?
21. Show that if f is a uniformly continuous function defined on (a,b), then f is bounded.